# Two different gauge-invariant models in the Lagrangian approach

Seung-Kook Kim†§, Yong-Wan Kim‡‖ and Young-Jai Park‡¶

† Department of Physics, Seonam University, Namwon, Chonbuk 590-170, Korea
‡ Department of Physics and Basic Science Research Institute, Sogang University, CPO Box 1142, Seoul 100-611, Korea

**Abstract.**   We show how to systematically derive the complete set of gauge transformations of two different types of gauge-invariant model, the chiral Schwinger and $CP^1$ with Chern–Simons term, in the Lagrangian formalism.

## 1. Introduction

The Hamiltonian embedding [1–6] of constrained systems has the drawback of not necessarily leading to a manifestly Lorentz covariant partition function. This problem could be avoided in the Lagrangian field–anti-field approach [7], which is based on an analysis of local symmetries of a Lagrangian. The establishment of the full, irreducible set of local symmetries of a Lagrangian thus plays a fundamental role in this formalism. In fact, these symmetries are often put, by hand, in the action while constructing the Lagrangian, and sometimes they are found by direct observation or trial and error. Moreover, it may be that for some complicated Lagrangians the full local symmetries cannot be seen directly. The systematic and exhaustive determination of gauge symmetry structure thus constitutes an integral part of the field–anti-field quantization program without the use of Dirac's Hamiltonian construction of the corresponding generators.

On the other hand, the Hamiltonian method of Batalin *et al* (BFT) [1] has been applied to second-class constrained systems [2, 3], which yield the strongly involutive first-class constraint algebra in an extended phase space. Recently, we have quantized other interesting models including the Proca models by using our improved BFT formalism [4–6].

In this paper, we will consider the Lagrangian approach of different types of gauge invariant systems, which have different constraint structure in the Lagrangian sense. According to the classification of constraints in [8], the A-type constraints are defined by functions without having velocities, while the B-type ones are a set of functions of velocity and coordinates. With this type of classification, we will show that the 'identically' vanishing parts of the successive evolutions for the stability of the constraints systematically generate the gauge symmetry of the system. Recently, Shirzad [9] tried to apply this Lagrangian formulation to the chiral Schwinger model (CSM) as well as the Schwinger

§ E-mail address: skandjh@chollian.net
‖ E-mail address: ywkim@physics3.sogang.ac.kr
¶ E-mail address: yjpark@ccs.sogang.ac.kr

model. However, he could not obtain the complete set of gauge transformations because he used the anomalous CSM, which is a gauge non-invariant second-class constraint system. Thus, we will briefly recapitulate the BFT Hamiltonian embedding [4, 10] of the gauge non-invariant CSM with $a > 1$ [11] in section 2, in order to show how one can systematically construct the gauge-invariant CSM. Then, we explicitly show how to derive the exact form of the gauge transformation of the first-class CSM model, which is of type B, making use of the iterated Lagrangian equations of motion [9, 12]. In section 3, we also derive the well known form of the gauge transformation of the gauge-invariant $CP^1$ model with Chern–Simons (CS) term [13–16], which is the case of mixed type A and B constraints. Our conclusions are given in section 4.

## 2. Chiral Schwinger model with Wess–Zumino (WZ) term

We first briefly recapitulate our previous BFT Hamiltonian embedding [4, 10] of the bosonized CSM model in the case of $a > 1$, whose dynamics are given by

$$S_{\text{CSM}} = \int d^2x \left[ -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} \partial_\mu \phi \partial^\mu \phi + eA_\nu(\eta^{\mu\nu} - \epsilon^{\mu\nu})\partial_\mu \phi + \frac{1}{2}ae^2 A_\mu A^\mu \right] \tag{1}$$

where $\eta^{\mu\nu} = \text{diag}(1, -1)$, $\epsilon^{01} = 1$, and $a$ is a regularization ambiguity [11]. The canonical momenta are given by $\pi_0 = 0, \pi_1 = F^{01}$, and $\pi_\phi = \dot{\phi} + e(A^0 + A^1)$. One then finds one primary [17] constraint $\Omega_1 \equiv \pi_0 \approx 0$ and one secondary constraint $\Omega_2 \equiv \partial^1 \pi_1 + e\pi_\phi + e\partial_1 \phi - e^2 A^1 + (a-1)e^2 A^0 \approx 0$, which is obtained by requiring the consistency of the primary constraint $\Omega_1$ with time evolution. These constraints fully form the second-class:

$$\Delta_{ij}(x, y) := \{\Omega_i(x), \Omega_j(y)\}$$
$$= -e^2(a-1)\epsilon_{ij}\delta(x-y) \tag{2}$$

and the canonical Hamiltonian $H_c$ is given by

$$H_c = \int dx \left[ \frac{1}{2}(\pi_1)^2 + \frac{1}{2}(\pi_\phi)^2 + \frac{1}{2}(\partial_1 \phi)^2 - e(\pi_\phi + \partial_1 \phi)(A^0 + A^1) \right.$$
$$\left. -A^0 \partial^1 \pi_1 - \frac{1}{2}ae^2\{(A^0)^2 - (A^1)^2\} + \frac{1}{2}e^2(A^0 + A^1)^2 \right]. \tag{3}$$

We now introduce auxiliary fields $\Phi^i$ in order to convert the second-class constraint $\Omega_i$ into first-class ones in an extended phase space. Following the BFT Hamiltonian formalism [1, 4–6, 10], we require these fields to satisfy

$$\{\Phi^i(x), \Phi^j(y)\} = \omega^{ij}(x, y) = \epsilon^{ij}\delta(x-y)$$
$$\{F, \Phi^i\} = 0 \tag{4}$$

where $F$ denotes the original variables, $(A^\mu, \pi_\mu, \phi, \pi_\phi)$, collectively. Strongly involutive constraints $\widetilde{\Omega}_i$ satisfying the requirement $\{\widetilde{\Omega}_i, \widetilde{\Omega}_j\} = 0$ as well as the boundary conditions, $\widetilde{\Omega}_i|_{\Phi^i=0} = \Omega_i$, can be obtained in power series as $\tilde{\Omega}_i = \sum_n \Omega_i^{(n)}$, where $\Omega_i^{(n)}$ is proportional to $(\Phi^j)^n$. The first-order correction terms in the infinite series are simply given by $\Omega_i^{(1)} = \int dy\, X_{ij}(x, y)\Phi^j(y)$. Then, the strongly involutive relation of $\widetilde{\Omega}_i$ provides the following condition:

$$\Delta_{ij}(x, y) + \int du\, dv\, X_{ik}(x, u)\omega^{k\ell}(u, v)X_{j\ell}(v, y) = 0 \tag{5}$$

from which we get a solution $X_{ij} = e\sqrt{a-1}\delta_{ij}\delta(x-y)$ in the case of choosing $\omega^{ij}$ as in equation (4). Now, making use of $\omega^{ij}$ and $X_{ij}$, we can easily obtain the strongly involutive constraints as $\widetilde{\Omega}_i = \Omega_i + e\sqrt{a-1}\Phi^i$.

On the other hand, corresponding to the original variables $F$, strongly involutive BFT variables $\widetilde{F} \equiv (\widetilde{A}^\mu, \widetilde{\pi}_\mu, \widetilde{\phi}, \widetilde{\pi}_\phi)$ such as $\{\widetilde{\Omega}_i, \widetilde{F}\} = 0$ are given by

$$\widetilde{A}^\mu = \left(A^0 + \frac{1}{e\sqrt{a-1}}\Phi^2, A^1 - \frac{1}{e\sqrt{a-1}}\partial_1\Phi^1\right)$$

$$\widetilde{\pi}_\mu = \left(\pi_0 + e\sqrt{a-1}\Phi^1, \pi_1 + \frac{e}{\sqrt{a-1}}\Phi^1\right)$$

$$\widetilde{\phi} = \phi - \frac{1}{\sqrt{a-1}}\Phi^1$$

$$\widetilde{\pi}_\phi = \pi_\phi - \frac{1}{\sqrt{a-1}}\partial_1\Phi^1. \tag{6}$$

Using these BFT fields, we can find the desired first-class Hamiltonian $\widetilde{H}$ from the canonical Hamiltonian $H_c$ as

$$\widetilde{H}(A^\mu, \pi_\nu, \phi, \pi_\phi; \Phi^i) = H_c(\widetilde{A}^\mu, \widetilde{\pi}_\nu, \widetilde{\phi}, \widetilde{\pi}_\phi)$$

$$= H_c(A^\mu, \pi_\nu, \phi, \pi_\phi) + \int dx \left[\frac{1}{2}(\partial_1\Phi^1)^2 + \frac{e^2}{2(a-1)}(\Phi^1)^2 + \frac{1}{2}(\Phi^2)^2\right.$$

$$\left. - \frac{1}{e\sqrt{a-1}}[e^2\pi_1 - e^2(a-1)\partial_1 A^1]\Phi^1 - \frac{1}{e\sqrt{a-1}}\Phi^2\widetilde{\Omega}_2\right] \tag{7}$$

which, by construction, is automatically strongly involutive, i.e. $\{\widetilde{\Omega}_i, \widetilde{H}\} = 0$.

It seems appropriate to comment on generators of local symmetry transformation in the Hamiltonian formulation, which are fully given by the first-class constraints. Defining the generators by

$$G := \sum_{\alpha=1}^{2} \int d^2x \, (-1)^{\alpha+1}\epsilon^\alpha(x)\widetilde{\Omega}_\alpha(x) \tag{8}$$

we have $(\delta A = \{A, G\})$

$$\begin{aligned}
\delta A^0 &= \epsilon^1 & \delta\pi_0 &= (a-1)e^2\epsilon^2 \\
\delta A^1 &= -\partial_1\epsilon^2 & \delta\pi_1 &= -e^2\epsilon^2 \\
\delta\phi &= -e\epsilon^2 & \delta\pi_\phi &= e\partial^1\epsilon^2 \\
\delta\theta &= -e\epsilon^2 & \delta\pi_\theta &= -(a-1)e\epsilon^1.
\end{aligned} \tag{9}$$

Without loss of any generality, we have inserted a factor $(-1)^{\alpha+1}$ in equation (8) in order to make the gauge transformation as usual, and also identified the new variables $\Phi^i$ as a canonically conjugate pair, $\Phi^i = (\sqrt{a-1}\theta, \pi_\theta/\sqrt{a-1})$, satisfying equation (4).

Now, we consider the partition function of the model in order to present the Lagrangian corresponding to $\widetilde{H}$. The starting phase space partition function is given by the Faddeev–Popov formula [18] as follows

$$Z = \int \mathcal{D}A^\mu \mathcal{D}\pi_\mu \mathcal{D}\phi \mathcal{D}\pi_\phi \mathcal{D}\theta \mathcal{D}\pi_\theta \prod_{i,j=1}^{2} \delta(\widetilde{\Omega}_i)\delta(\Gamma_j) \det|\{\widetilde{\Omega}_i, \Gamma_j\}|e^{iS'} \tag{10}$$

where

$$S' = \int d^2x \, (\pi_\mu \dot{A}^\mu + \pi_\phi \dot{\phi} + \pi_\theta \dot{\theta} - \widetilde{\mathcal{H}}') \tag{11}$$

and we have used an equivalent first-class Hamiltonian by adding a term proportional to the first-class constraint $\widetilde{\Omega}_2$:

$$\widetilde{H}' = \widetilde{H} + \int dx \, \frac{1}{(a-1)e} \pi_\theta \widetilde{\Omega}_2. \tag{12}$$

Note that the gauge fixing functions $\Gamma_j$ are chosen so that the determinant occurring in the functional measure is non-vanishing. The $\pi_0$ integration is trivially performed by exploiting the delta function $\delta(\widetilde{\Omega}_1) = \delta[\pi_0 + (a-1)e\theta]$. After exponentiating the remaining delta function $\delta(\widetilde{\Omega}_2) = \delta[\partial^1 \pi_1 + e\pi_\phi + e\partial_1\phi - e^2 A^1 + (a-1)e^2 A^0 + e\pi_\theta]$ in terms of a Fourier variable $\xi$ as $\delta(\widetilde{\Omega}_2) = \int \mathcal{D}\xi \exp[-i\int d^2x \, \xi \widetilde{\Omega}_2]$, transforming $A^0 \to A^0 + \xi$, and integrating the momentum variables $\pi_\phi$, $\pi_1$, and $\pi_\theta$, one could obtain the gauge invariant action up to a total divergence as follows

$$S = S_{\text{CSM}} + S_{\text{WZ}} \tag{13}$$

$$S_{WZ} = \int d^2x \left[ \frac{1}{2}(a-1)\partial_\mu\theta\partial^\mu\theta + e\partial_\mu\theta\{(a-1)\eta^{\mu\nu} + \epsilon^{\mu\nu}\}A_\nu \right] \tag{14}$$

where $S_{\text{WZ}}$ is the well known WZ term, which serves to cancel the gauge anomaly. The corresponding measure now reads

$$[\mathcal{D}\mu] = \mathcal{D}A^\mu \mathcal{D}\phi \mathcal{D}\theta \mathcal{D}\xi \prod_{\beta=1}^{2} \delta(\Gamma_\beta[A^0 + \xi, A^1, \phi_2\theta]) \det |\{\widetilde{\Omega}_\alpha, \Gamma_\beta\}|. \tag{15}$$

Now, we are ready to apply the Lagrangian approach [9, 12] to the gauge invariant action (13). The equations of motion are of the form

$$L_i^{(0)}(x) := \int dy \, [W_{ij}^{(0)}(x, y)\ddot{\varphi}^j(y) + \alpha_i^{(0)}(y)\delta(x - y)] = 0 \qquad i = 1, \dots, 4 \tag{16}$$

where $W_{ij}^{(0)}(x, y)$ is the Hessian matrix

$$W_{ij}^{(0)}(x, y) := \frac{\delta^2 \mathcal{L}}{\delta\dot{\varphi}^i(x)\delta\dot{\varphi}^j(y)}$$

$$= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & a-1 \end{pmatrix} \delta(x - y) = \widetilde{W}_{ij}^{(0)}\delta(x - y) \tag{17}$$

$$(\varphi^i)^{\text{T}}(x) := (A^0, A^1, \phi, \theta)(x) \tag{18}$$

$$(\alpha_i^{(0)})^{\text{T}}(x) := \int dy \left[ \frac{\partial^2 \mathcal{L}}{\partial\varphi^j(y)\partial\dot{\varphi}^i(x)} \partial\dot{\varphi}^j(y) \right] - \frac{\partial\mathcal{L}}{\partial\varphi^i(x)}$$

$$= (\alpha_{A^0}, \alpha_{A^1}, \alpha_\phi, \alpha_\theta)(x) \tag{19}$$

with

$$\alpha_{A^0} = \partial_1(\dot{A}^1 + \partial_1 A^0) - e\dot{\phi} - e\partial_1\phi - ae^2 A^0 - (a-1)e\dot{\theta} + e\partial_1\theta$$
$$\alpha_{A^1} = \partial_1\dot{A}^0 - e\dot{\phi} - e\partial_1\phi + ae^2 A^1 + e\dot{\theta} - (a-1)e\partial_1\theta$$
$$\alpha_\phi = e\dot{A}^0 + e\dot{A}^1 - \partial_1^2\phi + e\partial_1 A^0 + e\partial_1 A^1$$
$$\alpha_\theta = (a-1)e\dot{A}^0 - e\dot{A}^1 - (a-1)\partial_1^2\theta - e\partial_1 A^0 + (a-1)e\partial_1 A^1. \tag{20}$$

The Hessian matrix (17) is of rank three. Hence, there exists a 'zeroth generation' null eigenvector $\lambda^{(0)}(x, y)$ satisfying

$$\int dy \, \lambda_i^{(0)}(x, y)W_{ij}^{(0)}(y, z) = 0. \tag{21}$$

For simplicity, let us normalize it to have components

$$\lambda^{(0)}(x, y) = (1, 0, 0, 0)\delta(x - y).$$  (22)

Correspondingly, we have a 'zeroth generation' constraint

$$\Omega_1^{(0)}(x) = \int dy\, \lambda_i^{(0)}(x, y)L_i^{(0)}(y) = L_1^{(0)}(x) = \alpha_{A^0} = 0$$  (23)

in the Lagrangian sense.

Similar to the time stability condition of constraints in the Hamiltonian formalism, we now require the primary Lagrange constraint (23) to be independent of time. We thus need to add to the equations of motion (16) through the equation $\dot{\Omega}_1^{(0)} = 0$. Then, the resulting set of five equations may be summarized in the form of the set of 'first generation' equations, $L_{i_1}^{(1)}(x) = 0$, $i_1 = 1, \ldots, 5$, with

$$L_{i_1}^{(1)}(x) := \begin{cases} L_i^{(0)} & i = 1, \ldots, 4 \\ \dfrac{d}{dt}(\lambda_i^{(0)} L_i^{(0)}). \end{cases}$$  (24)

$L_{i_1}^{(1)}(x)$ can be written in the general form

$$L_{i_1}^{(1)}(x) := \int dy\, [W_{i_1 j}^{(1)}(x, y)\ddot{\varphi}^j(y) + \alpha_{i_1}^{(1)}(y)\delta(x - y)] = 0$$  (25)

where

$$W_{i_1 j}^{(1)}(x, y) = \begin{pmatrix} \widetilde{W}_{ij}^{(0)} \\ \hline 0 & \partial_1^x & -e & -(a-1)e \end{pmatrix}\delta(x - y)$$  (26)

and

$$(\alpha_{i_1}^{(1)})^T(x) = ((\alpha_i^{(0)})^T, \alpha_5^{(1)})(x)$$  (27)

with

$$\alpha_5^{(1)} = \partial_1^2 \dot{A}^0 - ae^2 \dot{A}^0 + e\partial_1 \dot{\phi} - e\partial_1 \dot{\theta}.$$  (28)

Next, let us repeat the previous analysis taking equation (25) as a starting point, and looking for solutions of a first generation null eigenvector of

$$\int dy\, \lambda_{i_1}^{(1)}(x, y)W_{i_1 j}^{(1)}(y, z) = 0.$$  (29)

Since $W_{i_1 j}^{(1)}(x, y)$ is still of rank three, there exists a null eigenvector, $\lambda^{(1)}(x, y)$, with the previous eigenvector extended as $\lambda^{(0)}(x, y) = (1, 0, 0, 0, 0)\delta(x - y)$. This $\lambda^{(1)}(x, y)$ is of the form $(0, -\partial_1^x, e, e, 1)v(x)\delta(x - y)$. We could thus choose it as

$$\lambda_{i_1}^{(1)}(x, y) = (0, -\partial_1^x, e, e, 1)\delta(x - y).$$  (30)

The associated constraint is found to vanish 'identically':

$$\Omega_2^{(1)}(x) = \int dy\, \lambda_{i_1}^{(1)}(x, y)L_{i_1}^{(1)}(y) = -\partial_1 \alpha_2^{(1)} + e\alpha_3^{(1)} + e\alpha_4^{(1)} + \alpha_5^{(1)} = 0.$$  (31)

Therefore, the algorithm ends at this stage.

The local symmetries of the action (13) are encoded in the identity (31). Recalling (16) and (25) we see that the identity (31) can be rewritten as follows

$$\Omega_2^{(1)}(x) = -\partial_1 L_2^{(0)} + eL_3^{(0)} + eL_4^{(0)} + \frac{d}{dt}L_1^{(0)} \equiv 0.$$  (32)

This result is a special case of a general theorem stating [8, 9] that the identities $\Omega_k^{(l)} \equiv 0$ can always be written in the form

$$\Omega_k^{(l)} := \sum_{s=0} \int \mathrm{d}y \left( (-1)^{s+1} \frac{\mathrm{d}^s}{\mathrm{d}t^s} \phi_k^{i(s)}(x, y) L_i^{(0)}(y) \right). \tag{33}$$

Then, for the gauge-invariant CSM, we have the following relations

$$\phi_2^{2(0)}(x, y) = \partial_1^x \delta(x - y)$$
$$\phi_2^{3(0)}(x, y) = -e\delta(x - y)$$
$$\phi_2^{4(0)}(x, y) = e\delta(x - y)$$
$$\phi_2^{1(1)}(x, y) = -\delta(x - y) \tag{34}$$

while all the others are vanishing.

On the other hand, it follows from general considerations [8, 9] that the action (13) is invariant under the transformation

$$\delta\varphi^i(y) := \sum_k \int \mathrm{d}x \, (\Lambda_k(x)\phi_k^{i(0)}(x, y) + \dot{\Lambda}_k(x)\phi_k^{i(1)}(x, y)). \tag{35}$$

For the CSM case this corresponds to the transformations

$$\delta A^\mu(x) = \partial^\mu \Lambda_2 \qquad \delta\phi(x) = -e\Lambda_2 \qquad \delta\theta(x) = -e\Lambda_2. \tag{36}$$

This is the set of symmetry transformations which is identical with the previous result (9) of the extended Hamiltonian formalism, when we set $\epsilon^1 = \partial^0 \epsilon^2$ and $\epsilon^2 = \Lambda_2$, similar to the Maxwell case [19]. As results, we have systematically derived the set of symmetry transformations starting from the Lagrangian of the gauge invariant CSM.

## 3. CP$^1$ model with CS term

The CP$^1$ model with CS term [13, 14], which is an archetype example of field theory and the constrained system of mixed type A and B, was considered by Polyakov who found Bose–Fermi statistics transmutation [20] in the model. Han [15] has analysed this CP$^1$ model by using the Dirac formalism together with the first-order Lagrangian method. We have recently analysed the CP$^1$ model with CS term by using fully the symplectic formalism [16].

Our starting Lagrangian for the gauge-invariant CP$^1$ model with CS term [13–16] to analyse in the Lagrangian approach is given by

$$\mathcal{L} = \frac{\kappa}{2\pi}\epsilon^{\mu\nu\rho}A_\mu \partial_\nu A_\rho + (\partial_\mu + \mathrm{i}A_\mu)z_a^*(\partial^\mu - \mathrm{i}A^\mu)z_a \qquad a = 1, 2 \tag{37}$$

with the CP$^1$ constraint

$$\Omega = |z_a|^2 - 1 = 0 \tag{38}$$

where the convention is $\eta^{\mu\nu} = \mathrm{diag}(1, -1, -1)$ and $\epsilon^{012} = +1$.

The equations of motion from the Lagrangian (37) can be written in the form

$$L_i^{(0)}(x) := \int \mathrm{d}^2x \, [W_{ij}^{(0)}(x, y)\ddot{\varphi}^j(y) + \alpha_i^{(0)}(y)\delta^2(x - y)] = 0 \qquad i = 1, \ldots, 7 \tag{39}$$

where $W_{ij}^{(0)}(x, y)$ is

$$W_{ij}^{(0)}(x, y) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix} \delta^2(x - y) = \widetilde{W}_{ij}^{(0)} \delta^2(x - y) \quad (40)$$

$$(\varphi^i)^{\mathrm{T}}(x) = (A^0, A^1, A^2, z_1, z_2, z_1^*, z_2^*)(x) \tag{41}$$

$$(\alpha_i^{(0)})^{\mathrm{T}}(x) = (\alpha_{A^0}, \alpha_{A^1}, \alpha_{A^2}, \alpha_{z_1}, \alpha_{z_2}, \alpha_{z_1^*}, \alpha_{z_2^*})(x) \tag{42}$$

with

$$\alpha_{A^0} = -\frac{\kappa}{\pi}\epsilon_{mn}\partial^m A^n - \mathrm{i}(z_a^*\dot{z}_a - \dot{z}_a^* z_a) - 2|z_a|^2 A^0$$

$$\alpha_{A^m} = -\frac{\kappa}{\pi}\epsilon_{nm}\dot{A}^n + \frac{\kappa}{\pi}\epsilon_{nm}\partial^n A^0 + \mathrm{i}(z_a^*\partial^m z_a - z_a\partial^m z_a^*) + 2|z_a|^2 A^m$$

$$\alpha_{z_a} = \mathrm{i}z_a^*\dot{A}^0 + 2\mathrm{i}A^\mu\partial_\mu z_a^* + \mathrm{i}z_a^*\partial_m A^m - z_a^* A_\mu A^\mu + \partial_m\partial^m z_a^*$$

$$\alpha_{z_a^*} = -\mathrm{i}z_a\dot{A}^0 - 2\mathrm{i}A^\mu\partial_\mu z_a - \mathrm{i}z_a\partial_m A^m - z_a A_\mu A^\mu + \partial_m\partial^m z_a. \tag{43}$$

Note that here $m = 1, 2$. The Hessian matrix (40) is of rank four. Hence, there exist three 'zeroth generation' null eigenvectors $\lambda_i^{(0)A}(x, y)$ satisfying

$$\int \mathrm{d}^2 y\, \lambda_i^{(0)A}(x, y) W_{ij}^{(0)}(y, z) = 0 \qquad A = 1, 2, 3. \tag{44}$$

We choose them to have components

$$\lambda_i^{(0)1}(x, y) = (1, 0, 0, 0, 0, 0, 0)\delta^2(x - y)$$

$$\lambda_i^{(0)2}(x, y) = (0, 1, 0, 0, 0, 0, 0)\delta^2(x - y)$$

$$\lambda_i^{(0)3}(x, y) = (0, 0, 1, 0, 0, 0, 0)\delta^2(x - y). \tag{45}$$

Correspondingly we have the 'zeroth generation' constraints, which are of B type

$$\Omega_k^{(0)} = \alpha_k = 0 \qquad k = 1, 2, 3. \tag{46}$$

On the other hand, in this $\mathrm{CP}^1$ case we have also one more constraint, i.e. $\mathrm{CP}^1$ constraint (38) which is of A type. Since the time derivative of $\Omega$ and $\mathrm{d}\Omega/\mathrm{d}t = z_a^*\dot{z}_a + z_a\dot{z}_a^*$ are independent of $\Omega_k$, we can obtain the following constraints

$$\Omega_k^{(0)} = \alpha_k$$

$$\Omega_4^{(0)} = \frac{\mathrm{d}\Omega}{\mathrm{d}t} = z_a^*\dot{z}_a + z_a\dot{z}_a^* \tag{47}$$

on the first stage of iteration.

Using the consistency condition for the constraints with time. We need to add the equation $\dot{\Omega}_{k'}^{(0)} = 0$, $(k' = 1, \ldots, 4)$ to the equation of motion (39). Hence we have the set of 'first generation' equations, $L_{i_1}^{(1)}(x) = 0$, $i_1 = 1, \ldots, 11$, as follows:

$$L_{i_1}^{(1)}(x) := \begin{cases} L_i^{(0)} & i_1 = 1, \ldots, 7 \\ \dfrac{\mathrm{d}}{\mathrm{d}t}(\lambda_i^{(0)A} L_i^{(0)}) & i_1 = 7 + A,\ A = 1, 2, 3 \\ \dfrac{\mathrm{d}}{\mathrm{d}t}(\Omega_4^{(0)}) & i_1 = 11. \end{cases} \tag{48}$$

$L_{i_1}^{(1)}(x)$ can be written in the general form

$$L_{i_1}^{(1)}(x) := \int d^2y \, [W_{i_1 j}^{(1)}(x, y)\ddot{\varphi}^j(y) + \alpha_{i_1}^{(1)}(y)\delta^2(x - y)] = 0 \qquad i_1 = 1, \dots, 11 \qquad (49)$$

where

$$W_{i_1 j}^{(1)}(x, y) = \left( \begin{array}{c} \widetilde{W}_{ij}^{(0)} \\ \hline \begin{array}{ccccccc} 0 & 0 & 0 & -iz_1^* & -iz_2^* & iz_1 & iz_2 \\ 0 & 0 & \kappa/\pi & 0 & 0 & 0 & 0 \\ 0 & -\kappa/\pi & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & z_1^* & z_2^* & z_1 & z_2 \end{array} \end{array} \right) \delta^2(x - y) \qquad (50)$$

and

$$(\alpha_{i_1}^{(1)})^{\mathrm{T}}(x) = ((\alpha_i^{(0)})^{\mathrm{T}}, \alpha_8^{(1)}, \alpha_9^{(1)}, \alpha_{10}^{(1)}, \alpha_{11}^{(1)})(x) \qquad (51)$$

with

$$\alpha_8^{(1)} = -2|z_a|^2 \dot{A}^0 - \frac{\kappa}{\pi}\epsilon_{mn}\partial^m \dot{A}^n - 2A^0 \frac{d}{dt}|z_a|^2$$

$$\alpha_9^{(1)} = -\frac{\kappa}{\pi}\partial^2 \dot{A}^0 + 2|z_a|^2 \dot{A}^1 + 2A^1 \frac{d}{dt}|z_a|^2 + i\frac{d}{dt}(z_a^*\partial^1 z_a - z_a\partial^1 z_a^*)$$

$$\alpha_{10}^{(1)} = \frac{\kappa}{\pi}\partial^1 \dot{A}^0 + 2|z_a|^2 \dot{A}^2 + 2A^2 \frac{d}{dt}|z_a|^2 + i\frac{d}{dt}(z_a^*\partial^2 z_a - z_a\partial^2 z_a^*)$$

$$\alpha_{11}^{(1)} = 2\dot{z}_a \dot{z}_a^* \qquad (52)$$

respectively.

Since $W_{ij}^{(1)}(x, y)$ is of rank six, there exist two 'first generation' null eigenvectors $\lambda^{(1)A}(x, y)$, $A = 1, 2$, with the previous three null eigenvectors extended as in section 2. Similarly using equation (29), these null eigenvectors are explicitly given by

$$\lambda_{i_1}^{(1)1}(x, y) = (0, 0, 0, -2iz_1, -2iz_2, 0, 0, 1, 0, 0, i)\delta^2(x - y)$$

$$\lambda_{i_1}^{(1)2}(x, y) = (0, 0, 0, 0, 0, -2iz_1^*, -2iz_2^*, -1, 0, 0, i)\delta^2(x - y). \qquad (53)$$

Associated with these eigenvectors we have new constraints

$$\Omega_1^{(1)} = 2i\dot{z}_a \dot{z}_a^* + 2A^0(z_a\dot{z}_a^* - z_a^*\dot{z}_a) - \frac{\kappa}{\pi}\epsilon_{mn}\partial^m \dot{A}^n + 4z_a A^m \partial_m z_a^* + 2|z_a|^2 \partial_m A^m$$

$$+ 2i|z_a|^2 A_\mu A^\mu - 2iz_a \partial_m \partial^m z_a^*$$

$$\Omega_2^{(1)} = 2i\dot{z}_a \dot{z}_a^* + 2A^0(z_a\dot{z}_a^* - z_a^*\dot{z}_a) + \frac{\kappa}{\pi}\epsilon_{mn}\partial^m \dot{A}^n - 4z_a^* A^m \partial_m z_a - 2|z_a|^2 \partial_m A^m$$

$$+ 2i|z_a|^2 A_\mu A^\mu - 2iz_a^* \partial_m \partial^m z_a. \qquad (54)$$

We now repeat the above procedure by using the consistency condition for the constraints with time, and obtain the 'second generation' equation, $L_{i_2}^{(2)}(x) = 0$, $i_2 = 1, \dots, 13$, with

$$L_{i_2}^{(2)}(x) := \begin{cases} L_i^{(0)} & i_1 = 1, \dots, 7 \\ \dfrac{d}{dt}(\lambda_i^{(0)A} L_i^{(0)}) & i_2 = 7 + A, \ A = 1, 2, 3 \\ \dfrac{d}{dt}(\Omega_4^{(0)}) & i_2 = 11 \\ \dfrac{d}{dt}(\lambda_{i_1}^{(1)B} L_{i_1}^{(1)}) & i_2 = 11 + B, \ B = 1, 2. \end{cases} \qquad (55)$$

The resulting complete set of equations is of the form

$$L_{i_2}^{(2)}(x) := \int d^2y \, [W_{i_2j}^{(2)}(x, y)\ddot{\varphi}^j(y) + \alpha_{i_2}^{(2)}(y)\delta^2(x - y)] = 0 \qquad i_2 = 1, \dots, 13 \qquad (56)$$

where

$$W_{i_2j}^{(2)}(x, y) = \left( \begin{array}{ccccccc} \multicolumn{7}{c}{\widetilde{W}_{ij}^{(0)}} \\ \hline 0 & 0 & 0 & -iz_1^* & -iz_2^* & iz_1 & iz_2 \\ 0 & 0 & \kappa/\pi & 0 & 0 & 0 & 0 \\ 0 & -\kappa/\pi & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & z_1^* & z_2^* & z_1 & z_2 \\ 0 & (\kappa/\pi)\partial^2 & -(\kappa/\pi)\partial^1 & W_{12,1}^{(2)} & W_{12,2}^{(2)} & W_{12,3}^{(2)} & W_{12,4}^{(2)} \\ 0 & -(\kappa/\pi)\partial^2 & (\kappa/\pi)\partial^1 & W_{13,1}^{(2)} & W_{13,2}^{(2)} & W_{13,3}^{(2)} & W_{13,4}^{(2)} \end{array} \right) \delta^2(x - y)$$

$$(57)$$

with

$$\begin{aligned} W_{12,k}^{(2)} &= 2(i\dot{z}_k^* - z_k^* A^0) = W_{13,k}^{(2)} \qquad k = 1, 2 \\ W_{12,l}^{(2)} &= 2(i\dot{z}_{l-2} - z_{l-2}A^0) = W_{13,l} \qquad l = 3, 4 \end{aligned} \qquad (58)$$

and

$$(\alpha_{i_2}^{(2)})^{\mathrm{T}}(x) = ((\alpha_{i_1}^{(1)})^{\mathrm{T}}, \alpha_{12}^{(2)}, \alpha_{13}^{(2)})(x) \qquad (59)$$

with

$$\begin{aligned} \alpha_{12}^{(2)} &= 2(z_a\dot{z}_a^* - \dot{z}_a z_a^*)\dot{A}^0 + 2|z_a|^2\partial_m \dot{A}^m + 4i|z_a|^2 A_\mu \dot{A}^\mu + 4\frac{d}{dt}(A^m z_a \partial_m z_a^*) \\ &\quad + 2\partial_m A^m \frac{d}{dt}|z_a|^2 + 2iA_\mu A^\mu \frac{d}{dt}|z_a|^2 - 2i\frac{d}{dt}(z_a \partial_m \partial^m z_a^*) \\ \alpha_{13}^{(2)} &= 2(z_a\dot{z}_a^* - z_a^*\dot{z}_a)\dot{A}^0 - 2|z_a|^2\partial_m \dot{A}^m + 4i|z_a|^2 A_\mu \dot{A}^\mu - 4\frac{d}{dt}(A^m z_a^* \partial_m z_a) \\ &\quad - 2\partial_m A^m \frac{d}{dt}|z_a|^2 + 2iA_\mu A^\mu \frac{d}{dt}|z_a|^2 - 2i\frac{d}{dt}(z_a^* \partial_m \partial^m z_a). \end{aligned} \qquad (60)$$

In addition to the previous null eigenvectors, we thus have a new null eigenvector

$$\lambda_{i_2}^{(2)}(x, y) = (0, 0, 0, 0, 0, 0, 0, 0, 2\partial_x^1, 2\partial_x^2, 0, 1, -1)\delta^2(x - y). \qquad (61)$$

The associated constraint is now found to vanish 'identically'

$$\Omega_1^{(2)}(x) = \int d^2y \, \lambda_{i_2}^{(2)}(x, y)L_{i_2}^{(2)}(y) = 2\partial^1\alpha_9^{(2)} + 2\partial^2\alpha_{10}^{(2)} + \alpha_{12}^{(2)} - \alpha_{13}^{(2)} = 0. \qquad (62)$$

The algorithm ends at this point.

The local symmetries of the action (37) are encoded in the identity (62). Using (39), (49), and (56), the identity (62) is equivalent to

$$\Omega_1^{(2)} = \frac{d}{dt}(\partial^1 L_2^{(0)} + \partial^2 L_3^{(0)} - iz_1 L_4^{(0)} - iz_2 L_5^{(0)} + iz_1^* L_6^{(0)} + iz_2^* L_7^{(0)}) + \frac{d^2}{dt^2}L_1^{(0)} \equiv 0. \qquad (63)$$

Comparing this with equation (33) for the gauge-invariant $CP^1$ model, we have the following relations

$$\phi_1^{2(1)}(x, y) = \partial_x^1 \delta^2(x - y) \qquad \phi_1^{3(1)}(x, y) = \partial_x^2 \delta^2(x - y)$$
$$\phi_1^{4(1)}(x, y) = -iz_1 \delta^2(x - y) \qquad \phi_1^{5(1)}(x, y) = -iz_2 \delta^2(x - y)$$
$$\phi_1^{6(1)}(x, y) = iz_1^* \delta^2(x - y) \qquad \phi_1^{7(1)}(x, y) = iz_2^* \delta^2(x - y)$$
$$\phi_2^{1(2)}(x, y) = \delta^2(x - y) \qquad\qquad\qquad\qquad\qquad (64)$$

while all the others are vanishing. Since the action (37) is invariant under the transformation

$$\delta\varphi^i(y) = \sum_k \int d^2x \, (\Lambda_k(x)\phi_k^{i(0)} + \dot{\Lambda}_k(x)\phi_k^{i(1)}(x, y) + \ddot{\Lambda}_k(x)\phi_k^{i(2)}(x, y)) \qquad (65)$$

the resulting symmetry transformations in the $CP^1$ model are finally obtained as

$$\delta A^\mu(x) = \partial^\mu \Lambda_1 \qquad \delta z_a(x) = -iz_a \Lambda_1 \qquad \delta z_a^*(x) = iz_a^* \Lambda_1 \qquad (66)$$

which are the well known gauge transformations of the corresponding model. As a result, we have systematically derived the set of symmetry transformations starting from the Lagrangian of the gauge-invariant $CP^1$ model with CS term.

## 4. Conclusion

In conclusion, we have considered the Lagrangian approach of two different types of gauge-invariant systems, the CSM having B-type constraints and the $CP^1$ model having mixed A- and B-type ones. We have first turned the anomalous CSM into a fully first-class constrained system, following the BFT method, and have shown how the symmetry transformation could be derived on a purely Lagrangian level, in particular without resorting to the Hamiltonian formulation. Furthermore, we have also systematically derived the well known symmetry transformation by analysing the $CP^1$ model, which is of different type from the CSM because this model imposes the $CP^1$ constraint by hand. As a result, we have shown that the Lagrangian approach could provide a systematic derivation of the symmetry transformation of a Lagrangian. We hope that the Lagrangian approach employed in our derivation will be of much interest for complicated Lagrangians whose full local symmetries cannot easily be extracted out, and also in the context of the field–anti-field formalism while keeping the manifestly Lorentz covariant partition function.

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## References

[1]  Batalin I A and Fradkin E S 1987 *Nucl. Phys.* B **279** 514
     Batalin I A and Fradkin E S 1986 *Phys. Lett.* **180B** 157
     Batalin I A and Tyutin I V 1991 *Int. J. Mod. Phys.* A **6** 3255
[2]  Banerjee R 1993 *Phys. Rev.* D **48** R5467
     Banerjee R *et al* 1994 *Phys. Rev.* D **49** 5438
[3]  Kim Y-W *et al* 1995 *Phys. Rev.* D **51** 2943
     Cha J-H *et al* 1995 *Z. Phys.* C **69** 175
[4]  Kim W T *et al* 1997 *J. Phys. G: Nucl. Part. Phys.* **23** 325

[5] Kim Y-W and Rothe K D 1998 *Nucl. Phys.* B **510** 511
    Kim Y-W *et al* 1998 *J. Phys. G: Nucl. Part. Phys.* **24** 953
[6] Kim Y-W *et al* 1997 *Int. J. Mod. Phys.* A **12** 4217
    Park M-I and Park Y-J 1998 *Int. J. Mod. Phys.* A **13** 2179
[7] Batalin I A and Vilkovsky G A 1981 *Phys. Lett.* **102B** 27
    Batalin I A and Vilkovsky G A 1983 *Phys. Rev.* D **28** 2567
    Batalin I A and Vilkovsky G A 1984 *Phys. Rev.* D **30** 508 (Erratum)
    Gomis J *et al* 1995 *Phys. Rep.* **259** 1
[8] Sudarshan E C G and Mukunda N 1974 *Classical Dynamics: A Modern Perspective* (New York: Wiley)
[9] Shirzad A 1998 *J. Phys. A: Math. Gen.* **31** 2747
[10] Kim Y-W *et al* 1992 *Phys. Rev.* D **46** 4574
    Kim S-K *et al* 1995 *J. Korean Phys. Soc.* **28** 128
[11] Jackiw R and Rajaraman R 1985 *Phys. Rev. Lett.* **54** 1219
    Jackiw R and Rajaraman R 1985 *Phys. Rev. Lett.* **54** 2060 (Erratum)
    Falck N K and Kramer G 1987 *Ann. Phys.* **176** 330
    Shizuya K 1988 *Phys. Lett.* **213B** 298
    Harada K 1990 *Phys. Rev. Lett.* **64** 139
[12] Kim Y-W and Rothe K D 1998 *Int. J. Mod. Phys.* A **13** 4183
[13] Deser S *et al* 1982 *Ann. Phys., NY* **140** 372
[14] Panigrashi P K *et al* 1988 *Phys. Rev. Lett.* **61** 2827
[15] Han C 1993 *Phys. Rev.* D **47** 5522
[16] Kim Y-W *et al* 1995 *J. Korean Phys. Soc.* **28** 773
[17] Dirac P A M 1964 *Lectures on Quantum Mechanics (Belfer Graduate School)* (New York: Yeshiba University Press)
[18] Faddeev L D and Popov V N 1967 *Phys. Lett.* **25B** 29
[19] As is well known, symmetry transformations of the Lagrangian generally imply restrictions on the symmetry transformations of the total Hamiltonian. See also, Henneaux M and Teitelboim C 1992 *Quantization of Gauge Systems* (Princeton, NJ: Princeton University Press)
[20] Polyakov A M 1988 *Mod. Phys. Lett.* A **3** 325